GENERAL SOLUTION TO THE JET FLOW PAST A WEDGE

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UDC 533.6.01

An exact general solution to the jet flow (in particular, of incompressible fluid) past a wedge (and flat plate) has been obtained in this paper for the case when the stagnation point is at the apex of the wedge. The stream function and the relations establishing the connection between the wedge parameters, flow, and the wedge location have been obtained. As an example, the general solution is used for solving the flow past a wedge at the surface of an unbounded flow which is the generalization of one of the problems posed by N. E. Zhukovskii. It is shown that the general solution contains solutions for all the various earlier particular cases of the problem.

1. The flow past a wedge (l_1 and l_2 are the lengths of the side walls, 2α is the wedge angle) by a subsonic fluid jet with velocity V_0, width H, and mass flow rate Q, at an angle μ to the wedge axis (x axis) is split at the stagnation point 0 located at the wedge apex into two jets of width H_1 and H_2, with mass flow rates Q_1 and Q_2, respectively, at angles δ and γ to the wedge axis (Fig. 1). This classical problem was first studied for an incompressible flow by Kotel'nikov [1], who gave the solution to the particular case $Q_1 = Q_2$. In [2], Zhukovskii indicated that his method gives a solution even when $Q_1 \neq Q_2$, though the solution itself was not given. The general solution to the problem ($Q_1 \neq Q_2$) for compressible subsonic flow is given in [3] which also contains solution to the incompressible flow as a particular case. In [4] the solution for a liquid jet is mentioned but the final results are not given. In the second edition [5] only reference to [1, 2] is given. Later, solutions (with errors) for gas flow are given in [6, 7]. Since these errors play havoc with solution to the classical problem, there is sense in returning to the general solution [3], all the more because there is a need for a significant improvement. This solution has the form

$$\psi = \sum_{n=1}^{\infty} a_n \frac{Z_{nq/2}(\tau)}{Z_{nq/2}(\tau_0)} \sin n\theta'; \tag{1.1}$$

$$a_n = \frac{2Q}{\pi n} (\cos n\mu' - K_2 \cos n\gamma' - K_1 \cos n\delta'); \tag{1.2}$$

$$l_1 = \frac{Q}{V_0(1-\tau_0)^{\beta}} \left\{ \frac{1}{\sin 2\alpha} [\cos(\mu+\alpha) - K_2 \cos(\gamma+\alpha) - K_1 \cos(\delta+\alpha)] + \right. \tag{1.3}$$

$$\left. + \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} nq}{n^2 q^2 - 1} (\cos n\mu' - K_2 \cos n\gamma' - K_1 \cos n\delta') X_{nq/2}(\tau_0) \right\};$$

$$l_2 = \frac{Q}{V_0(1-\tau_0)^{\beta}} \left\{ \frac{1}{\sin 2\alpha} [\cos(\mu-\alpha) - K_2 \cos(\gamma-\alpha) - K_1 \cos(\delta-\alpha)] \right. \tag{1.4}$$

$$\left. + \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} nq}{n^2 q^2 - 1} (\cos n\mu'' - K_2 \cos n\gamma'' - K_1 \cos n\delta'') X_{nq/2}(\tau_0) \right\};$$

$$K_1 + K_2 = 1, \quad \sin q\mu = K_2 \sin q\gamma + K_1 \sin q\delta; \tag{1.5}$$

$$R = QV_0[1 - K_2 \cos(\mu - \gamma) - K_1 \cos(\mu - \delta)], \tag{1.6}$$

$$R_1 = \frac{QV_0}{\sin 2\alpha} [\cos(\mu+\alpha) - K_2 \cos(\gamma+\alpha) - K_1 \cos(\delta+\alpha)],$$

$$R_2 = \frac{QV_0}{\sin 2\alpha} [\cos(\mu-\alpha) - K_2 \cos(\gamma-\alpha) - K_1 \cos(\delta-\alpha)]$$

$$(K_1 = Q_1/Q,\ K_2 = Q_2/Q,\ q = \pi/2\alpha,\ \theta' = q\theta + \pi/2,\ \theta'' = q\theta - \pi/2).$$

Vologda. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 98-103, May-June, 1983. Original article submitted March 3, 1982.

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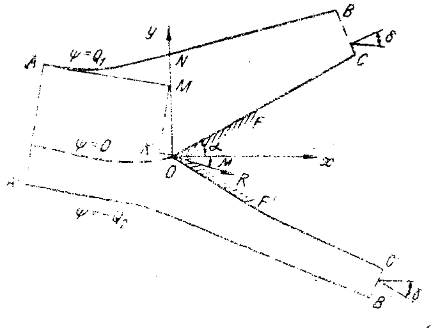

Fig. 1

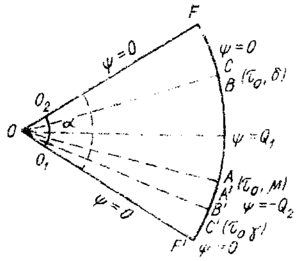

Fig. 2

Here ψ is the stream function; τ, θ, Chaplygin variables [5, 8, and 9]; R_1 and R_2, pressures at the wedge side walls; R, wedge resistance; τ_0, value of τ at the surface; $\rho^0 = 1$, stagnation density; $\beta = 1/(\chi - 1)$, χ, polytropic index.

The correctness of (1.1) and (1.2) can be ascertained directly by contour integration around the contour that confines the flow in the hodograph plane (Fig. 2). Equations (1.3), (1.4), and (1.6) can be obtained by integrating the transformation equations from the hodograph plane to the physical plane along the wedge walls or along lines of constant velocity.

2. The second equation in (1.5) has been derived in [3] only for $q \leqslant 1$ (transformed wedge and flat plate). We show that it satisfies all q.

In an infinitely small neighborhood of the point 0 from (1.1), we have

$$\psi \sim a_n \frac{Z_{nq/2}(\tau)}{Z_{nq/2}(\tau_0)} \sin n\theta', \tag{2.1}$$

where n is the number of the first nonzero term in the series (1.1). For incompressible fluid

$$\psi \sim a_n(\tau/\tau_0)^{nq/2} \sin n\theta' = a_n(V/V_0)^{nq} \sin n\theta'. \tag{2.2}$$

In going around the point 0 along an infinitely small arc O_1O_2 in the hodograph plane (τ, θ) (see Fig. 2) the change in argument equals 2α, whereas for the same path in the region of complex potential $W = \varphi + i\psi$ the change in argument is 2π. Hence it follows [5] that in the infinitely small neighborhood of the point 0

$$W \sim \mathrm{const}\left(\frac{V}{V_0} e^{i\theta}\right)^{\pi/2}. \tag{2.3}$$

A comparison of (2.2) and (2.3) gives n = 2. The same result also follows directly from (2.1). Actually, while going along O_1O_2 the line $\psi = 0$ (see Fig. 2) must be crossed, i.e., $\sin n\theta'$ should become zero only once in the interval from $\theta' = 0$ ($\theta = -\pi/2$) to $\theta' = \pi$ ($\theta = \pi/2$). It is possible to observe that this is attainable only for n = 2.

Thus, series (1.1) begins from the second term and $a_1 = 0$, and for any q the following equality must be satisfied

$$\cos \mu' - K_2 \cos \gamma' - K_1 \cos \delta' = 0 \tag{2.4}$$

and summation in (1.1), (1.3), and (1.4) starts from n = 2. Equation (2.4) is, however, equivalent to the second equation in (1.5) which is satisfied, thus, for all q. This fact was not noticed for nearly a quarter of a century [3, 6, and 7].

From (1.5) we find

$$K_1 = (\sin q\mu - \sin q\gamma)/(\sin q\delta - \sin q\gamma),$$
$$K_2 = (\sin q\mu - \sin q\delta)/(\sin q\gamma - \sin q\delta). \tag{2.5}$$

Substituting (2.5) in (1.3) and (1.4), we have two equations for the unknown parameters γ and δ. After obtaining γ and δ from (2.5), we find K_1 and K_2, and thereby from (1.2), (1.1) the stream function ψ is determined and from it all other flow parameters are obtained. Thus, (1.1)-(1.5) is the complete solution. It is valid for all q, including flat plate ($q = 1$). Here, when $n = 1$, in order to avoid indeterminacy of the type 0/0 in (1.3) and (1.4) it is necessary to switch over to the limit $q \to 1$. This can be easily carried out though, in order to avoid writing two parallel series of equations, we shall sum up from $n = 1$, keeping in view the need for the limiting approach mentioned above for the flat plate; for the wedge, in view of (2.4) this does not lead to any changes, since $\alpha_1 = 0$.

If $\beta = 0$, $X_{nq/2}(\tau_0) = 1$, in (1.1)-(1.6)

$$\frac{Z_{nq/2}(\tau)}{Z_{nq/2}(\tau_0)} = \left(\frac{\tau}{\tau_0}\right)^{nq/2} = \left(\frac{V}{V_0}\right)^{nq},$$

then the solution is obtained for the incompressible flow problem which was also not completely solved earlier. Here, for stream function ψ the series is summed up in terms of elementary functions, and the sum of the series in (1.3) and (1.4) can be expressed in terms of proper integrals, which in the case where q is a rational number (i.e., when the wedge angle can be measured in terms of π) can also be expressed in terms of elementary functions [10, 11].

3. Since relations (1.1)-(1.5) completely determine the flow, they make it possible to find any flow parameter, in particular, the depth of immersion $h = MK$ of the wedge vertex in the jet flow with respect to its upper level at infinity (see Fig. 1). We shall give this solution since errors were committed in [6, 7] while deriving similar relations.

It is sufficient to derive the main relation for $L = OM$ since $h = L \cos \mu$. Let AM be the asymptote of the upper surface in the jet and its equation has the form

$$y = \operatorname{tg} \mu x + L,$$

where L is the coordinate of the point of intersection of the asymptote with the x axis.

At infinitely far sections of the jets

$$\lim_{\theta\to\delta} (x_{FC} - x_{AB}) = H_1 \sin \delta, \; \lim_{\theta\to\delta} (y_{AB} - y_{FC}) = H_1 \cos \delta, \quad \lim_{\theta\to\mu} (y_{AB} - \operatorname{tg} \mu x_{AB}) = L. \tag{3.1}$$

For free surfaces of the jets we have [5, 8]

$$y = y_0 + \int_{\theta_0}^{\theta} \frac{\sin\theta}{V_0} \frac{2\tau_0}{(1-\tau_0)^\beta} \left(\frac{\partial\psi}{\partial\tau}\right)_{\tau=\tau_0} d\theta, \quad x = x_0 + \int_{\theta_0}^{\theta} \frac{\cos\theta}{V_0} \frac{2\tau_0}{(1-\tau_0)^\beta} \left(\frac{\partial\psi}{\partial\tau}\right)_{\tau=\tau_0} d\theta, \tag{3.2}$$

where x_0, y_0 are the coordinates of the point D on the jet; θ_0 is the angle of inclination of the velocity at this point.

Using the expression for stream function (1.1), (1.2), and (3.2) we write equations for the surface AB and FC, representing the point D for the jet AB by the point $N(0, y_1)$ at angle $\theta_0 = \theta_1$ and for the jet FC by the point $F(l_1 \cos\alpha, l_1 \sin \alpha)$ at $\theta_0 = \alpha$. The relations obtained are substituted in (3.1) from which the combination $L - H_1 \cos \delta - \tan \mu H_1$ and δ is obtained. We have

$$\frac{\pi}{2q}\left[\frac{L\cos\mu}{H} - \frac{l_1 \sin(\alpha-\mu)}{H} - K_1 \cos(\delta-\mu)\right] = \sum_{n=1}^{\infty} (\cos n\mu' - K_2 \cos n\gamma' - K_1 \cos n\delta') X_{nq/2}(\tau_0) \int_{\mu}^{\alpha} \sin(\mu-\theta) \sin n\theta' d\theta. \tag{3.3}$$

Computing the integrals and using the expression (1.3) for l_1, we find the depth h:

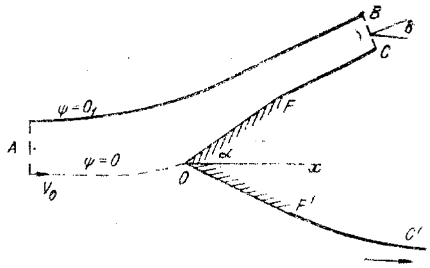

Fig. 3

$$\frac{h}{H} - K_1 \cos(\delta - \mu) = \frac{\sin(\alpha - \mu)}{\sin 2\alpha} [\cos(\mu + \alpha) - K_2 \cos(\gamma + \alpha) - $$
$$- K_1 \cos(\delta + \alpha)] + \sum_{n=1}^{\infty} \frac{\sin n\mu'}{n^2 q^2 - 1} (\cos n\mu' - K_2 \cos n\gamma' - K_1 \cos n\delta') X_{nq/2}(\tau_0). \quad (3.4)$$

The presence of three equations (1.3), (1.4), and (3.4) for two unknown parameters γ and δ shows that for the given wedge geometry l_1, l_2, and 2α, flow parameters V_0, H and the flow structure (see Fig. 1), the wedge orientation is not arbitrary, but only for a given orientation of the wedge axis μ the depth of immersion h should be completely determined and vice versa.

If, however, we specify K_1 and K_2, then the angle μ and depth h are determined from (1.3)-(1.5), (3.4).

Consequently, the solution [1] with the condition

$$K_1 = K_2 = 1/2 \quad (3.5)$$

determines the flow, though it is possible that it is a special case.

The flat plate is a special case. For it the unknowns are l_1, l_2, δ, and γ which are determined by Eqs. (1.3), (1.4), and (3.4) and $l_1 + l_2 = 2l$ ($2l$ is the plate length), here h and μ are independent.

In deriving an analog of Eq. (3.4), incorrect relations were obtained in [6] which were again found in a somewhat different form in [7] (Eqs. (17.13) of that paper). Analyzing these equations, the author of [7] maintains that in any case $Q_1 = Q_2$, i.e., (3.5) is always satisfied.

This conclusion is wrong since it is made from incorrect equation (17.13) in deriving which the author [7] ignored that the point A is a singular point of the flow and the solution used by him for each concrete τ is a Fourier series for which Dirichlet theorem [12] is valid.

4. As an example, we use the more general solution to the flow past a wedge located at the surface of an unbounded gas flow when the free stream is parallel to the wedge axis ($\mu = 0$) (Fig. 3).

We shall start from the solution to (1.1)-(1.6) for the jet with finite flux Q. For $\mu = 0$, from (1.5)

$$Q_2/Q_1 = -\sin q\delta / \sin q\gamma. \quad (4.1)$$

Taking into account (4.1), we find

$$\psi = \frac{2Q_1}{\pi} \sum_{n=2}^{\infty} \frac{1}{n} \left[\cos\frac{n\pi}{2} - \cos n\delta' - \sin q\delta \frac{\cos\frac{n\pi}{2} - \cos n\gamma'}{\sin q\gamma} \right] \frac{Z_{nq/2}(\tau)}{Z_{nq/2}(\tau_0)} \sin n\theta'; \quad (4.2)$$

$$l_1 + l_2 = \frac{Q_1}{V_0 (1 - \tau_0)^\beta} \left\{ \frac{1}{\sin\alpha} \left[1 - \cos\delta - \sin q\delta \frac{1 - \cos\gamma}{\sin q\gamma} \right] + \right.$$

$$+\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4nq}{4n^2q^2-1}\left[1-\cos 2nq\delta-\sin q\delta\frac{1-\cos 2nq\gamma}{\sin q\gamma}\right]X_{nq}(\tau_0)\Bigg\}; \tag{4.3}$$

$$R = Q_1V_0\left[1-\cos\delta-\sin q\delta\frac{1-\cos\gamma}{\sin q\gamma}\right]. \tag{4.4}$$

$Q_2 = \infty$ corresponds to this problem (Fig. 3). As seen from (4.1), as $Q_2 \to \infty$ (Q_1 = const) γ should go to zero. Allowing this limiting case in (4.2)-(4.4) we find for this problem

$$\psi = \frac{2Q_1}{\pi}\sum_{n=2}^{\infty}\frac{1}{n}\left[\cos\frac{n\pi}{2}-\cos n\delta'-\sin q\delta\sin\frac{n\pi}{2}n\right]\frac{Z_{nq/2}(\tau)}{Z_{nq/2}(\tau_0)}\sin n\theta'; \tag{4.5}$$

$$l_1+l_2 = \frac{Q_1}{V_0(1-\tau_0)^{\beta}}\left[\frac{1-\cos\delta}{\sin\alpha}+\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4nq}{4n^2q^2-1}(1-\cos 2nq\delta)X_{nq}(\tau_0)\right]; \tag{4.6}$$

$$R = Q_1V_0(1-\cos\delta). \tag{4.7}$$

Let us apply the general solution to (1.1)-(1.6) for the case of symmetric flow past a wedge ($\mu = 0$, $\gamma = -\delta$, $l_1 = l_2 = l$, $K_1 = K_2 = 1/2$). We have

$$\psi = \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\cos 2nq\delta)\frac{Z_{nq}(\tau)}{Z_{nq}(\tau_0)}\sin 2nq\theta; \tag{4.8}$$

$$2l = \frac{Q}{V_0(1-\tau_0)^{\beta}}\left[\frac{1-\cos\delta}{\sin\alpha}+\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4nq}{4n^2q^2-1}(1-\cos 2nq\delta)X_{nq}(\tau_0)\right]; \tag{4.9}$$

$$R = QV_0(1-\cos\delta). \tag{4.10}$$

These equations for q = 1 (flat plate) are obtained in [8].

A comparison of (4.6), (4.7) with (4.9), (4.10) leads to the result: The drag of the wedge placed at the surface of unbounded flow parallel to the wedge axis ($\mu = 0$) and the jet impinging across the wedge with a flux Q_1 is the same as the symmetric flow past an equal sided wedge by a fluid jet with the same flux Q_1, if the sum of the side wall lengths of these wedges is identical ($l_1 + l_2 = 2l$).

This result appears even more clearly for the plate: infinite flow with velocity V_0 impinging along the edges of a flat plate placed normal to it and jet impingement through it with a flux Q_1 causes the same pressure on it as the jet impinging on the center of the plate at a velocity V_0 and flux Q_1.

This interesting result was established for incompressible flow by N. E. Zhukovskii who studied the flat plate problem. Relations (4.9) and (4.10) for q = 1 also lead to the equations [2] even for an incompressible flow.

The chosen problem for which $K_1 = 0$, $K_2 = 1$ is a good contradictory example to establish the complete satisfaction of (3.5).

5. Putting $\delta = \alpha$ in the general solution (1.1)-(1.6) we get $l_1 = \infty$ and obtain the solution to the flow past a half plane with a break at the point O. In addition, putting $\alpha = \pi/2$ here we get the solution for the flow past a half plane without the break [13].

If in the general solution $\delta = \alpha$, $\gamma = -\alpha$, we get $l_1 = \infty$, $l_2 = \infty$ and come to the solution for the flow past a plane wall with a bend; for $\alpha = \pi/2$, we get the solution for the flow past the plane wall without inflection [14, 15].

It is also possible to obtain the solution for the flow past a wedge by an unbounded fluid using the limiting approach ($Q \to \infty$). This is done in [3] for the flat plate case.

Thus, the general solution (1.1)-(1.6), (3.4) covers all particular cases of the problem.

LITERATURE CITED

1. A. P. Kotel'nikov, "Fluid jet pressure on a wedge," in: Collected Transactions of the Phys. Math. Section in Kazan [in Russian], No. 8, Kazan (1889).
2. N. E. Zhukovskii, "Transformation of Kirchhoff method," in: Complete Works [in Russian], Vol. 3, ONTI, Moscow—Leningrad (1936).
3. V. I. Troshin, "Solution to some jet problems in subsonic gasdynamics," Author's Abstract of Candidate's Dissertation, Physical-Mathematical Sciences, Moscow State University (1959).
4. M. I. Gurevich, The Theory of Jets in an Ideal Fluid, Pergamon (1966).
5. M. I. Gurevich, Theory of Jets in Ideal Fluids, Academic Press (1965).
6. M. N. Makeev, "Some problems in theory of jets in subsonic gasdynamics," Author's Abstract of Doctoral Dissertation, Physical-Mathematical Sciences University of Kazan (1968).
7. E. D. Tomilov, Subsonic Plane Jet Flows [in Russian], Nauka, Novosibirsk (1980).
8. S. A. Chaplygin, Fluid Jets [in Russian], GTTI, Moscow—Leningrad (1949).
9. L. I. Sedov, Plane Problems in Hydrodynamics and Aerodynamics [in Russian], Nauka, Moscow (1980).
10. T. Bromwich, An Introduction to Theory of Infinite Series, [Russian translation], Leningrad (1955).
11. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series [in Russian], Nauka, Moscow (1981).
12. A. Zigmund, Trigonometric Series, Cambridge Univ. Press.
13. T. S. Solomakhova, "Plane jet flow on a semiinfinite screen," Vestn. Mosk. Gos. Univ. Ser. 1, No. 1 (1963).
14. G. I. Nazarov, "Oblique impingement of two dimensional gas jet on a plane wall," Prikl. Math. Mekh., 21, No. 3 (1957).
15. N. A. Slezkin, "Plane gas jet impingement on infinite plate," Prikl. Math. Mekh., 16, No. 2 (1952).

EXPERIMENTAL STUDY OF THE INTERACTION OF A PAIR OF HYPERSONIC JETS

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UDC 533.6.011.8

Jet interaction in certain types of vacuum pumps appreciably affects the evacuating capacity and the limiting vacuum. However, there have been no goal-oriented studies on this phenomenon applicable to vacuum pumps [1]. Jet interaction studies [2-4] carried out for fairly high Reynolds numbers characterizing the viscous effects, and small values of nozzle spacing do not relate directly to the operating conditions of vacuum pumps. The interaction of a pair of adjacent jets is similar to the interaction of a single jet with a surface parallel to the jet axis, without friction. The formulation of such studies is of interest in solving problems associated with the effects of jet strength on the surrounding elements under the conditions of vacuum. The present paper is devoted to the experimental study of the influence of viscous effects on the density distribution in the symmetry plane of a pair of parallel, strongly underexpanded hypersonic jets under conditions similar to vacuum pumps.

The flow structure in the interaction region of jets issuing into a heated space is determined by Mach number M at the nozzle section, ratio of specific heats γ, stagnation parameters of the fluid jet p_0 and T_0, pressure in the surrounding medium p_k, outside temperature T_k, Reynolds number Re_* based on the parameters at the throat section, and the geometric parameters: h, the distance between the nozzles, d_*, the throat diameter. In order to determine the location of the geometric surfaces it is sufficient to use the ratio of

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Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 104-107, May-June, 1983. Original article submitted May 10, 1982.

0021-8944/83/2403-0379$07.50